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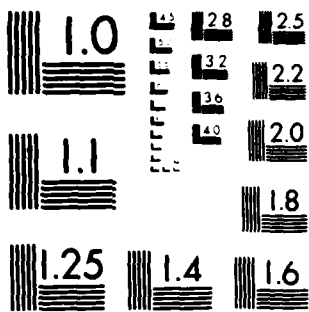
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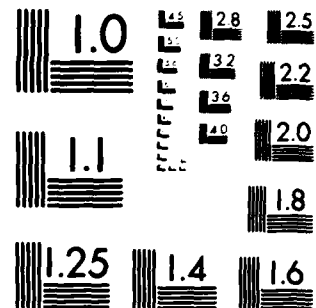
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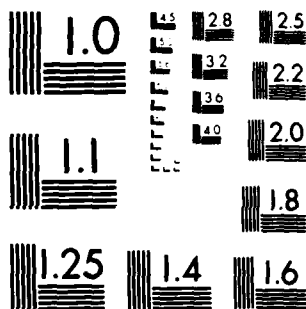
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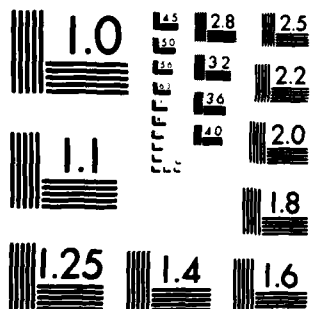
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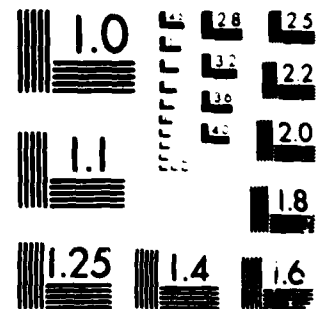
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MULTIPLICATIVITY OF  $\ell_p$  NORMS FOR MATRICES

To Professor Alexander Ostrowski on his 90th birthday

ADA121149

Moshe Goldberg\*  
Department of Mathematics  
Technion - Israel Institute  
of Technology  
Haifa 32000, Israel

E.G. Straus\*\*  
Department of Mathematics  
University of California  
Los Angeles, California 90024

and

Institute for the Interdisciplinary  
Applications of Algebra  
and Combinatorics  
University of California  
Santa Barbara, California 93106

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# ABSTRACT

The  $\ell_p$  norm and the  $\ell_p$  operator-norm of an  $m \times n$  complex matrix  $A = (\alpha_{ij})$  are given by

$$|A|_p = \left( \sum_{i,j} |\alpha_{ij}|^p \right)^{1/p}$$

and

$$\|A\|_p = \max \{ |Ax|_p : x \in \mathbb{C}^n, |x|_p = 1 \},$$

respectively. The main purpose of this paper is to investigate the multiplicativity of the  $\ell_p$  norms and their relation to the  $\ell_p$  operator-norms.

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# 1. Introduction and Statement of Main Results.

For  $1 \leq p \leq \infty$ , the  $\ell_p$  norm of an  $m \times n$  matrix  $A = (\alpha_{ij}) \in \mathbb{C}_{m \times n}$  is defined as

$$(1.1) \quad |A|_p = \left( \sum_{i=1}^m \sum_{j=1}^n |\alpha_{ij}|^p \right)^{1/p}$$

where for the case  $p = \infty$  (which need not be treated separately) we have, of course,

$$|A|_\infty = \lim_{p \rightarrow \infty} |A|_p = \max_{i,j} |\alpha_{ij}|.$$

That is,  $|\cdot|_p$  on  $\mathbb{C}_{m \times n}$  is simply the ordinary  $\ell_p$  norm on  $\mathbb{C}^{mn}$ .

These  $\ell_p$  norms must be distinguished from the  $\ell_p$  operator-norms on  $\mathbb{C}_{m \times n}$ ,

$$(1.2) \quad \|A\|_p = \max\{ |Ax|_p : x \in \mathbb{C}^n, |x|_p = 1 \}.$$

Ostrowski [4], investigated some sub-multiplicativity properties of the  $\ell_p$  norms in (1.1), including the following:

**THEOREM 1.1** [4, Theorem 7.] If  $1 \leq p \leq 2$  and if  $A, B$  are rectangular matrices so that the product  $AB$  exists, then

$$|AB|_p \leq |A|_p |B|_p.$$

THEOREM 1.2 [4, Theorem 8.] If  $p, q$  satisfy  $1 \leq p \leq 2 \leq q$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , and if  $A, B$  are rectangular matrices so that  $AB$  exists, then

$$|AB|_q \leq |A|_q |B|_p,$$

$$|AB|_q \leq |A|_p |B|_q.$$

The results in Theorem 1.2 are analogous to Hölder's Inequality.

While Ostrowski proved that for  $1 \leq q < 2 < p$  the inequalities in Theorems 1.1 and 1.2 may fail to hold, we are able to generalize his results as follows:

THEOREM 1.3. If  $1 \leq q \leq 2 \leq p$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , and if  $A \in \mathbb{C}_{m \times k}$ ,  $B \in \mathbb{C}_{k \times n}$ , then

$$|AB|_p \leq k^{1-2/p} |A|_p |B|_p.$$

THEOREM 1.4. If  $1 \leq q \leq 2 \leq p$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , and  $A \in \mathbb{C}_{m \times k}$ ,  $B \in \mathbb{C}_{k \times n}$ , then

$$|AB|_q \leq n^{1-2/q} |A|_p |B|_q,$$

$$|AB|_q \leq m^{1-2/q} |A|_q |B|_p.$$

A unified proof for Theorems 1.1-1.4 is given in Section 2.

Note that if  $A = a = (\alpha_1, \dots, \alpha_k)$  is a row vector and  $B = b^* = (\beta_1, \dots, \beta_k)^*$  is a column vector (\* denoting the adjoint), then  $AB$  is the usual inner product  $(a, b)$  on  $\mathbb{C}^k$ ; hence Theorems 1.1 and 1.3 give in this case the two inequalities

$$|(a,b)| \leq |a|_p |b|_p, \quad 1 \leq p \leq 2,$$

$$|(a,b)| \leq k^{1-2/p} |a|_p |b|_p, \quad p \geq 2;$$

and Theorems 1.2 and 1.4 yield the Hölder Inequality

$$(1.3) \quad |(a,b)| = \left| \sum_{i=1}^k \alpha_i \bar{\beta}_i \right| \leq \left( \sum_{i=1}^k |\alpha_i|^p \right)^{1/p} \left( \sum_{i=1}^k |\beta_i|^q \right)^{1/q} = |a|_p |b|_q,$$

$$p \geq 1, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

A norm  $N$  on  $\mathcal{C}_{n \times n}$  is commonly called (sub-) multiplicative if in addition to the ordinary norm properties

$$N(A) > 0, \quad A \neq 0,$$

$$N(\lambda A) = |\lambda| N(A), \quad \lambda \in \mathbb{C},$$

$$N(A+B) \leq N(A) + N(B),$$

we also have,

$$N(AB) \leq N(A)N(B)$$

for all  $A, B \in \mathcal{C}_{n \times n}$ .

Obviously, if  $N$  is a norm on  $\mathcal{C}_{n \times n}$  and  $\mu > 0$  is a fixed constant, then  $\mu N$  is a norm on  $\mathcal{C}_{n \times n}$  too. This new norm may or may not be multiplicative. If it is, we call  $\mu$  a multiplicativity factor for  $N$ . That is,  $\mu$  is a multiplicativity factor for  $N$  if and only if

$$N(AB) \leq \mu N(A)N(B), \quad \forall A, B \in \mathcal{C}_{n \times n}.$$

If  $\mu_0$  is a multiplicativity factor for  $N$  then clearly, so is any  $\mu$  with  $\mu \geq \mu_0$ . In fact we proved more:

THEOREM 1.5 [1, Theorem 4.]

(i) If  $N$  is a norm on  $\mathbb{C}_{n \times n}$  then  $N$  has multiplicativity factors.\*

(ii) A constant  $\mu > 0$  is a multiplicativity factor for  $N$  if and only if

$$\mu \geq \mu_N \equiv \max\{N(AB) : N(A) = N(B) = 1\}.$$

Thus,  $\mu_N$  is the optimal (smallest) multiplicativity factor for  $N$  if and only if

$$N(AB) \leq \mu_N N(A)N(B) \quad A, B \in \mathbb{C}_{n \times n}$$

with equality for some nonzero matrices  $A = A_0, B = B_0$ .

We observe now that matrices  $A, B$  whose upper left entry is 1 and all other entries vanish, yield equality in Theorems 1.1 and 1.2. Similarly, matrices  $A, B$  all whose entries are 1, give equality in Theorems 1.3 and 1.4. Hence, Theorems 1.1-1.5 immediately provide the following result for our  $\ell_p$  norms on square matrices:

COROLLARY 1.1. The optimal multiplicativity factor  $\mu_p \equiv \mu_{|\cdot|_p}$  for the norm  $|\cdot|_p$  on  $\mathbb{C}_{n \times n}$  satisfies

$$(1.4) \quad \mu_p \equiv \mu_p(n) = \begin{cases} 1 & , \quad 1 \leq p \leq 2, \\ n^{1-2/p} & , \quad p \geq 2. \end{cases}$$

If we define now the multiplicative norm

$$(1.5) \quad M_p(A) \equiv \mu_p \cdot |A|_p, \quad A \in \mathbb{C}_{n \times n},$$

---

\* This is not always the case for norms on infinite dimensional algebras; see Section 2 of [2].



then Theorems 1.1-1.4 for square matrices, together with Corollary 1.1, can be restated as,

COROLLARY 1.2. For all  $A, B \in \mathbb{C}_{n \times n}$  and all  $p, q$  with  
 $p \geq 1, \frac{1}{p} + \frac{1}{q} = 1,$  we have

$$M_p(AB) \leq M_p(A)M_p(B),$$

$$M_p(AB) \leq M_p(A)M_q(B),$$

$$M_p(AB) \leq M_q(A)M_p(B),$$

where in general, these inequalities are best possible.

The following relations between the  $\ell_p$  norms in (1.1) and the  $\ell_p$  operator-norms in (1.2) are special cases of Theorems 1.1-1.4, as will follow from Theorem 2.1 in the next section:

THEOREM 1.6. If  $A \in \mathbb{C}_{m \times n}$  and  $p > 1, \frac{1}{p} + \frac{1}{q} = 1,$  then

$$\|A\|_p \leq \mu_p(n) |A|_p,$$

$$\|A\|_p \leq \mu_q(m) |A|_q.$$

We remark that if  $1 \leq p \leq 2$  then Theorem 1.6 implies

$$\|A\|_p \leq |A|_p$$

which is meaningful also for bounded linear operators on infinite  $\ell_p$  spaces. If  $p > 2$ , we get

$$\|A\|_p \leq |A|_q$$

which again may be meaningful in the infinite dimensional case.

The main tool in proving Theorem 1.1-1.4 in Section 2 is the following lemma which seems to be of independent interest.

LEMMA 1.1 (Main Lemma.) For every vector  $x \in \mathbb{C}^n$  and  
 $1 \leq p \leq q \leq \infty$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , we have

$$(1.6) \quad |x|_q \leq |x|_p \leq n^{1/p-1/q} |x|_q.$$

Our proofs in Section 2 also make use of the mixed  $\ell_{p,q}$  norms on  $\mathbb{C}_{m \times n}$ , introduced by Ostrowski [4] as

$$(1.7) \quad |A|_{p,q} = \left( \sum_{j=1}^n \left( \sum_{i=1}^m |\alpha_{i,j}|^p \right)^{q/p} \right)^{1/q}.$$

The main result concerning these mixed norms is given in Theorem 2.1.

We emphasize that each one of the inequalities established in this paper becomes an equality, either when we deal with matrices (including vectors) with a single entry 1 in the upper left hand corner and all other entries 0; or when we deal with matrices all of whose entries are 1. Thus, none of our inequalities in this paper can be improved.

## 2. Further Results and Proofs.

Proof of the Main Lemma. The fact that  $|x|_p$  is an increasing function of  $p$ ,  $p \geq 1$ , is well known (e.g. [3]); and this is the statement of the left inequality in (1.6).

We write now Hölder's Inequality in (1.3) as

$$\left| \sum_{i=1}^n \alpha_i \beta_i \right| \leq \left( \sum_{i=1}^n |\alpha_i|^s \right)^{1/s} \left( \sum_{i=1}^n |\beta_i|^t \right)^{1/t}, \quad s \geq 1, \quad \frac{1}{s} + \frac{1}{t} = 1.$$

Thus, for  $s = q/p$ ,  $t = q/(q-p)$  and any  $x = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ , we have

$$\begin{aligned} |x|^p &= \sum_{i=1}^n |\xi_i|^p = \sum_{i=1}^n |\xi_i|^{p \cdot 1} \leq \left( \sum_{i=1}^n |\xi_i|^q \right)^{p/q} \left( \sum_{i=1}^n 1 \right)^{1-p/q} \\ &= n^{p(1/p-1/q)} |x|_q^p, \end{aligned}$$

and the right inequality in (1.6) follows.  $\square$

Recall now the definition of the norm  $M_p(A)$  in (1.5) for square matrices. In an analogous way, for rectangular matrices  $A \in \mathbb{C}_{m \times n}$  we define two different multiples of  $|A|_p$  as follows:

$$(2.1) \quad M'_p(A) = \mu_p(m) |A|_p,$$

$$(2.2) \quad M''_p(A) = \mu_p(n) |A|_p,$$

where  $\mu_p(n)$  is given in (1.4).

Clearly, for square matrices we have

$$M_p(A) = M'_p(A) = M''_p(A).$$

On the other hand, viewing from now on a vector  $x \in \mathbb{C}^n$  as an  $n \times 1$  matrix, our notation becomes

$$(2.3) \quad M'_p(x) = \mu_p(n) |x|_p;$$

hence we can reformulate our Main Lemma in the following compact way:

LEMMA 2.1. If  $p \geq 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ , then

$$(2.4) \quad |x|_p \leq M'_q(x), \quad \forall \quad x \in \mathbb{C}^n.$$

Proof. If  $p \leq q$ , then (2.4) is the right inequality of (1.6).  
If  $p \geq q$ , then  $q \leq 2$ , so  $M'_q(x) = |x|_q$ , and (2.4) becomes the left inequality of (1.6).  $\square$

Having the definitions in (2.1), (2.2), we prove next:

LEMMA 2.2. If  $A = (\alpha_{ij}) \in \mathbb{C}_{m \times n}$ , and  $p \geq 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , then

$$(2.5) \quad |A|_{p,q} \leq M'_q(A),$$

$$(2.6) \quad |A|_{p,q} \leq M''_p(A).$$

Proof. Denoting the columns of  $A$  by  $a_1, \dots, a_n$ , Lemma 2.1 and (2.3) yield

$$\begin{aligned} |A|_{p,q} &= \left( |a_1|_p, \dots, |a_n|_p \right)_q \leq \left( M'_q(a_1), \dots, M'_q(a_n) \right)_q \\ &= \mu_q(m) \left( |a_1|_q, \dots, |a_n|_q \right)_q = \mu_q(m) |A|_q = M'_q(A), \end{aligned}$$

and we have (2.5).

For (2.6) we use again Lemma 2.1 and (2.3) to obtain

$$\begin{aligned} |A|_{p,q} &= |(|a_1|_p, \dots, |a_n|_p)|_q \leq M'_p(|a_1|_p, \dots, |a_n|_p) \\ &= \mu_p(n) |(|a_1|_p, \dots, |a_n|_p)|_p = \mu_p(n) |A|_p = M''_p(A). \quad \square \end{aligned}$$

In [4] Ostrowski gives the following proof to the following lemma.

LEMMA 2.3 [4, Section 35.] For  $p \geq 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , and matrices  $A, B$  for which  $AB$  exists, we have

$$(2.7) \quad |AB|_p \leq |A|_p |B|_{q,p},$$

$$(2.8) \quad |AB|_p \leq |B|_p |A^T|_{q,p},$$

where  $T$  denotes the transpose.

Proof. Let  $A \in \mathbb{C}_{n \times k}$ ,  $B \in \mathbb{C}_{k \times n}$ , and set

$$\gamma_{ij} = \sum_{\ell=1}^k \alpha_{i\ell} \beta_{\ell j}.$$

Then by Hölder's Inequality we have,

$$|\gamma_{ij}|^p \leq \left| \sum_{\ell=1}^k \alpha_{i\ell} \beta_{\ell j} \right|^p \leq \sum_{\ell=1}^k |\alpha_{i\ell}|^p \left( \sum_{\ell=1}^k |\beta_{\ell j}|^q \right)^{p/q}.$$

So

$$|AB|_p^p = \sum_{i=1}^m \sum_{j=1}^n |\gamma_{ij}|^p \leq |A|_p^p \sum_{j=1}^n \left( \sum_{\ell=1}^k |\beta_{\ell j}|^q \right)^{p/q} = |A|_p^p |B|_{q,p}^p,$$

and (2.7) is established.

For (2.8) we exchange the roles of  $p$  and  $q$  to obtain

$$|Y_{ij}|^p \leq \sum_{\ell=1}^k |\beta_{\ell j}|^p \left( \sum_{\ell=1}^k |\alpha_{i\ell}|^q \right)^{p/q};$$

hence

$$|AB|_p^p \leq |B|_p^p \sum_{i=1}^m \left( \sum_{\ell=1}^k |\alpha_{i\ell}|^q \right)^{p/q} = |B|_p^p |A^T|_{q,p}^p,$$

and the lemma follows.  $\square$

With the help of Lemmas 2.2 and 2.3 we can now prove the main result of this section:

**THEOREM 2.1.** If  $p \geq 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , and if  $A, B$  are rectangular matrices so that  $AB$  exists, then

$$(2.9) \quad M'_p(AB) \leq M'_p(A)M'_p(B),$$

$$(2.10) \quad M''_p(AB) \leq M''_p(A)M''_p(B),$$

$$(2.11) \quad M'_p(AB) \leq M'_p(A)M''_q(B),$$

$$(2.12) \quad M''_p(AB) \leq M'_q(A)M''_p(B),$$

where  $M'$  and  $M''$  are defined in (2.1), (2.2).

Proof. Let  $A \in \mathbb{C}_{m \times k}$  and  $B \in \mathbb{C}_{k \times n}$ . Then, by (2.7) and (2.5) we have

$$|AB|_p \leq |A|_p |B|_{q,p} \leq |A|_p M'_p(B),$$

and multiplying both sides by  $\mu_p(m)$  yields (2.9).

Since  $M'_p(A^T) = M''_p(A)$  then by (2.8) and (2.5),

$$|AB|_p \leq |B|_p |A^T|_{q,p} \leq |B|_p M'_p(A^T) = |B|_p M''_p(A),$$

so multiplying by  $\mu_p(n)$  gives (2.10).

Next, we use (2.7) and (2.6) to obtain

$$|AB|_p \leq |A|_p |B|_{q,p} \leq |A|_p M''_q(B),$$

and multiplying by  $\mu_q(m)$  gives (2.11).

Finally, by (2.8) and (2.6),

$$|AB|_p \leq |B|_p |A^T|_{q,p} \leq |B|_p M''_q(A^T) = M'_q(A) |B|_p,$$

so multiplying by  $\mu_p(n)$  yields (2.12).  $\square$

We observe now that Theorems 1.1-1.4 are merely restatements of (2.9)-(2.12).

Proof of Theorem 1.6. If  $A \in \mathbb{C}_{m \times n}$  and  $x \in \mathbb{C}^n$ , we think as before of  $x$  as an  $n \times 1$  matrix; hence,

$$|Ax|_p = M''_p(Ax), \quad M''_p(x) = |x|_p.$$

So by (2.10),

$$\begin{aligned} \|A\|_p &= \max_{|x|_p=1} |Ax|_p = \max_{|x|_p=1} M''_p(Ax) \leq M''_p(A) \max_{|x|_p=1} M''_p(x) \\ &= M''_p(A) = \mu_p(n) |A|_p, \end{aligned}$$

and we get the first part of the theorem.

Similarly, if  $A$  and  $x$  are as above, then by (2.12),

$$\begin{aligned} \|A\|_p &= \max_{\|x\|_p=1} \|Ax\|_p = \max_{\|x\|_p=1} M'_p(Ax) \leq M'_q(A) \max_{\|x\|_p=1} M''_p(x) \\ &= M'_q(A) = \mu_q(m) \|A\|_q, \end{aligned}$$

and the proof is complete.  $\square$

Inequalities for products of more than two matrices which are treated by Ostrowski in [4], can be extended in a manner entirely analogous to our results in this paper.

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